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WELL-POSEDNESS IN M-ULTRADIFFERENTIABLE SPACES FOR WEAKLY HYPERBOLIC CAUCHY PROBLEMS WITH HÖLDER CONTINUOUS COEFFICIENTS

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ABSTRACT

In this article, we demonstrate the weakly hyperbolic Cauchy problem under Hölder's regularity of a coefficient depending on time in the context of **M**-ultradifferentiable well-posedness. We find an equivalent condition for well-posedness within the framework of Gevrey regularity, ensuring well-posedness in a class of M-ultradifferentiable functions, dependent on the associated function of the sequence **M**. viscoelastic materials.

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I. INTRODUCTION

Consider the problem:

$$\begin{cases} u_{tt} = \sum_{i,j=1}^{N} a_{ij}(t) u_{x_i x_j}, & \text{on} \quad \mathbb{R}^N \times]0, L] \\ u = g_0, \quad u_t = g_1 & \text{when} \quad t = 0. \end{cases}$$

The well-posedness of this equation has been analyzed in several studies, such as [1], [2]. These works demonstrate existence and uniqueness results, even in cases where the coefficients a_{ij} are not smooth, particularly when they are integrable over [0, L], assuming the strict hyperbolicity condition holds:

$$\sum a_{ij}(t)\zeta_i \zeta_j \geq \lambda_0 |\zeta|^2 \quad (\lambda_0 > 0).$$

Of course, solutions have same smoothness when g_0, and g_1 are entire functions. But under the weakly hyperbolicity condition

$$\sum a_{ij}(t)\zeta_i \zeta_j \geq 0, \quad \forall \zeta \in \mathbb{R}^N,$$

The well-posedness of problem (1.1) is no longer guaranteed, even if the coefficients belong to C^{∞}, as discussed [3]. Additionally, Colombini, Jannelli, and Spagnolo [4] demonstrated that well-posedness holds under *s*-Gevrey regularity, provided the condition

$$1 \leq s < 1 + \frac{k+\alpha}{2},$$

Under the weakly hyperbolicity condition, whenever the coefficients $a_{ij}(t)$ belong to Hölder space $C^{k,\alpha}([0,L])$, with k integer ≥ 0 and $0 \leq \alpha < 1$. To broaden the analysis of well-posedness from the s -Gevrey class to M-ultradifferentiable functions of class M under weakly hyperbolic condition where the coefficients $a_{ij}(t)$ belong to Hölder space $C^{k,\alpha}([0,L])$, which request equivalent condition of (1.3). In different way from the work of Colombini [5] which based of weight function, we find another condition dependent with associated function of sequence M taken in consideration the work of Braun, Meise, and Taylor [6]. It is well known that if $M = \left(p!^{s}\right)_{p \geq 0}$, then we recover the s -Gevrey space([7], [8]). At the duality level, the work of Garetto, and Ruzhansky [9], [10] based on the fundamental condition (1.3) studied the well-posedness in Gevrey ultradistributions classes and above all in Generalized Gevrey ultradistributions classes [11]. To examine the Cauchy problem in the setting of generalized Roumieu ultradistributions remains an unresolved question, as discussed in [12]. Therefore, it is natural to search for an equivalent condition.

II. PRELIMINAIRES AND NOTATION

In this section, we present the essential notations, definitions, and key estimates needed for the formulation of our main results. The definitions and concepts presented are inspired by the foundational works of [13], [14], [7]. Throughout this paper, we denote by $M = \left(M_{p}\right)_{p \in \mathbb{N}_{+}}$ a sequence of positive numbers.

II.1 M-ULTRADIFFERENTIABLE FUNCTIONS OF CLASS M

Recall the following properties:

- Logarithmic Convexity Property:

$$M_{p-1}M_{p+1} \geq M_{p}^{2} \quad \forall p \geq 1 \quad (LCP)$$

- Ultra differentiation Property:

$$M_{p+q} \leq M_{p}M_{q}BA^{p+q}, \quad \forall p,q \geq 0, \quad \text{for some } B > 0, A > 0. \quad (UP)$$

- Differentiation Property:

$$\exists B > 0, \exists A > 0, \quad BA^{p}M_{p} \geq M_{p+1}, \quad \forall p \geq 0 \quad (DP)$$

- Non-quasi-analyticity

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_{p}} < +\infty \qquad (NQA)$$

II.1.1 DEFINITION

(M-Ultradifferentiable function). Let $\omega \subset \mathbb{R}^{N}$. A function g belonging to $C^{\infty}(\omega)$ is considered M-ultradifferentiable class (M_{p}) if, for every *k* compact subset of ω, the following condition holds:

$$\exists A > 0, \exists B > 0, \quad \sup_{x \in K} \left|D^{\beta} g(x)\right| \leq BA^{|\beta|}M_{|\beta|}$$

For all $\beta \in \mathbb{N}^{N}$, where the length of β is given by

$$|\beta| = \beta_{1} + \beta_{2} + \cdots + \beta_{N}.$$

The set of these functions is denoted by $\mathrm{E}^{\{M_{p}\}}(\omega)$ and is known as the M -ultradifferentiable functions of Roumieu. Similarly, a function $g \in C^{\infty}(\omega)$ is in M-ultradifferentiable class (M_{p}) if, for every K compact subset of ω, the following condition holds:

$\forall A>0, \exists B>0, \quad \sup_{x\in K}\left|D^{\beta}g(x)\right| \leq BA^{|\beta|}M_{|\beta|}$ for all $\beta \in \mathbb{N}^N$. The set of these functions is denoted by $\mathcal{E}^{(M_p)}(\omega)$ and is known as the M -ultradifferentiable functions of Beurling. To simplify notation, we use $*$ to refer to either (M_p) or $\{M_p\}$.

II.1.2 Definition

Denoted by

$$\mathcal{D}^*(\omega) = \mathcal{E}^*(\omega) \cap \mathcal{D}(\omega),$$

The M -ultradifferentiable space with compact support. This space is nontrivial precisely when the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfies condition (**NQA**). We substitute $\mathcal{E}^*(\mathbb{R}^N)$ by $\mathcal{E}^*$.

II.1.3 Definition

Given a sequence $M=(M_p)$, its associated function $M(x)$ is defined for $x>0$ by:

$$M(x) = \sup_p \ln \frac{x^p}{M_p}.$$

II.1.4 Example

The Gevrey associated function is given by:

$$M(x) = x^{\frac{1}{s}}, \quad s>0.$$

II.1.5 Theorem

(Paley-Wiener Theorem). Let M be its associated function. Assume (**LCP**), (**DP**) and (**NQA**). Let $K \subset \mathbb{R}^N$ be compact convex set. An entire function $\hat{g}_0 : \mathbb{R}^n \to \mathbb{R}$ satisfies $\left(\exists A\in\mathbb{R}^+\right)\left(\exists B\in\mathbb{R}^+\right)\left(\forall \zeta \in \mathbb{R}^N : |\hat{g}_0(\zeta)| \leq B\exp\left(-M(|\zeta|/A)\right)\right)$ if and only if $\exists g_0 \in \mathcal{D}^{(M_p)}(K)$ such that $\hat{g}_0$ is the Fourier transform of g_0. Analogously, an entire function $\hat{g}_0 : \mathbb{R}^N \to \mathbb{R}$ satisfies $\left(\forall A\in\mathbb{R}^+\right)\left(\exists B\in\mathbb{R}^+\right)\left(\forall \zeta \in \mathbb{R}^N : |\hat{g}_0(\zeta)| \leq B\exp\left(-M(|\zeta|/A)\right)\right)$ if and only if $\exists g_0 \in \mathcal{D}^{(M_p)}(K)$.

II.2 HŐLDER SPACES

We denote by $C^{\alpha,k}([0,L])$, where $k $ is a nonnegative integer and $0 \leq \alpha < 1$, the real Hölde space, associate with norm given by

$$\|\phi\|_{C^{\alpha,k}} = \sum_{h=0}^{k} \sup_{[0,L]} |\phi^{(h)}| + \sup_{t\neq s} \frac{|\phi^{(k)}(t)-\phi^{(k)}(s)|}{|t-s|^{\alpha}}.$$

The proof of the following lemma is available in [4] and is necessary for proving our theorem.

II.3 LAMMA

Let I be a real compact interval, and h be a function of $C^{\alpha,k}(I)$. If $h \geq 0$ on I. Then

$$\left\|\left(h^{1/(k+\alpha)}\right)'\right\|_{L^1(I)}^{k+\alpha} \leq C_{k,\alpha}(I)\|h\|_{C^{\alpha,k}(I)},$$

where $C_{k,\alpha}(I)$ is positive.

III. THE RESULTS

III.1 THEOREM

Let M satisfies (**LCP**), (**DP**) and (**NQA**). Consider the problem

$$\begin{cases} u_{tt} = \sum_{i,j=1}^{N} a_{ij}(t) u_{x_i x_j} \\ u(x,0) = g_0(x), \\ u_t(x,0) = g_1(x) \end{cases}$$

On $\mathbb{R}^N \times [0,L]$, assuming that

$$\sum a_{ij}(t)\zeta_i\zeta_j \geq 0, \quad \forall \zeta \in \mathbb{R}^N, \forall t$$

And

$$a_{ij} \in C^{\alpha,k}([0,L]), \quad k \text{ integer } > 0, 0 \leq \alpha < 1.$$

For every g_0 and g_1 in E^* and if

$$M\left(\rho/h\right) \leq \rho^{2/(2+k+\alpha)}, \quad \forall \rho \in \mathbb{R}_+^*,$$

Then problem admits one and only one solution $u \in C^2\left([0,L],\mathrm{E}^*\right)$.

III.2 REMARK

In Gevrey class the condition (3.2) is equivalent to (1.3).

Proof Let $v(\zeta,t)$ denote the Fourier transform of $u(x,t)$, given by

$$v(\zeta,t) = \int_{\mathbb{R}^N} u(x,t)e^{-ix\zeta}dx, \quad \zeta \in \mathbb{R}^N.$$

According to the Paley-Wiener theorem (see [7]), it is sufficient to verify that for $h \in \mathbb{R}^+$, there exists $C_0 \in \mathbb{R}^+$ Where, for every $(\zeta,t) \in \mathbb{R}^N \times [0,L]$, it follows that:

$$|v(\zeta,t)| \leq C_0 \exp\left(-M\left(\frac{|\zeta|}{h}\right)\right).$$

Consequently, from (3.3), the solution *u* of (3.1) is contained in $C^2([0,L],\mathrm{D}^*(\mathbb{R}^N))$. By Fourier transform, (3.1) becomes

$$v'' + \left(a(t)\zeta, \zeta\right)v = 0, \quad t \in [0,L],$$

Where

$$\left(a(t)\zeta, \zeta\right) = \sum a_{ij}(t)\zeta_i\zeta_j.$$

We now construct an appropriate approximation of $a(t)$ using a family $\{a_\varepsilon(t)\}_{\varepsilon>0}$ of C^1 strictly positive quadratic forms. For each $\varepsilon > 0$, we define the corresponding ε-approximate energy of *u*.

$$E_\varepsilon(\zeta,t) = \left(a_\varepsilon(t)\zeta,\zeta\right)|v|^2 + |v'|^2$$

we have

$$E_\varepsilon'(\zeta,t) = \left(a_\varepsilon'\zeta,\zeta\right)|v|^2 + 2\left(a_\varepsilon\zeta,\zeta\right)\mathrm{Re}\left(v\bar{v}'\right) + 2\,\mathrm{Re}\left(\bar{v}'v''\right)$$

by (3.4) we have

$$E_\varepsilon' \leq \left|\left(a_\varepsilon'\zeta,\zeta\right)\right||v|^2 + 2\left|\left(\left(a_e - a\right)\zeta,\zeta\right)\right||v||v'|$$

which mean

$$E_\varepsilon' \leq \frac{\left|\left(a_\varepsilon'\zeta,\zeta\right)\right|}{\left(a_\varepsilon\zeta,\zeta\right)}E_\varepsilon + \frac{\left|\left(\left(a_\varepsilon - a\right)\zeta,\zeta\right)\right|}{\left(a_\varepsilon\zeta,\zeta\right)^{\frac{1}{2}}}E_\varepsilon$$

By Gronwall lemma $\forall t \in [0,L]$

$$E_\varepsilon(\zeta,t) \leq E_\varepsilon(\zeta,0)\exp\left(\int_0^L \frac{\left|\left(a_\varepsilon'\zeta,\zeta\right)\right|}{\left(a_\varepsilon\zeta,\zeta\right)}ds + \int_0^L \frac{\left|\left(\left(a_s - a\right)\zeta,\zeta\right)\right|}{\left(a_\varepsilon\zeta,\zeta\right)^3}ds\right)$$

by considering separately the case in which $k\geqslant1$ and the case $k = 0$ at first, we assume

$$a_{\varepsilon} = a + \varepsilon I$$

where I represent the identity matrix. Consequently, we clearly obtain

$$(a_{\varepsilon}\zeta,\zeta) \geq (a_{\varepsilon}\zeta,\zeta)^{1-1/(k+\alpha)}\left(\varepsilon|\zeta|^{2}\right)^{1/(k+\alpha)}$$

and

$$\frac{\left|\left((a_{\varepsilon}-a)\zeta,\zeta\right)\right|}{(a_{\varepsilon}\zeta,\zeta)^{\frac{1}{2}}} \le \sqrt{\varepsilon}\,|\zeta|$$

Alternatively, by applying Lemma 1 with $h(t) = (a(t)\zeta,\zeta)$, we derive

$$\int_{0}^{L}\frac{\left(a_{\varepsilon}^{'}\zeta,\zeta\right)}{(a_{\varepsilon}\zeta,\zeta)^{1-1/(k+\alpha)}}ds \le C_{k,\alpha}(L)\,\|a\|_{C^{\alpha,k}}^{1/(k+\alpha)}|\zeta|^{2/(k+\alpha)}$$

Introducing (3.6), (3.7) and (3.8) in (3.5) we obtain then the estimate

$$E_{\varepsilon}(\zeta,t) \le E_{\varepsilon}(\zeta,0)\exp\left[C_{1}\left(\varepsilon^{-1/(k+\alpha)}+\sqrt{\varepsilon}\,|\zeta|\right)\right]$$

where $C_1 \in \mathbb{R}^{+}$ and we put

$$E(\zeta,t) = |\zeta|^{2}|v(\zeta,t)|^{2} + |v^{'}(\zeta,t)|^{2}.$$

We observe the a_{ij} are bounded on $[0,L]$, such that exist $\Lambda > 0$,

$$\sum a_{ij}(t)\zeta_{i}\zeta_{j} \le \Lambda|\zeta|^{2}, \quad \forall\zeta \in \mathbb{R}^{N}, \forall t$$

then

$$\varepsilon E(\zeta,t) \le E_{\varepsilon}(\zeta,t) \le (1+\Lambda)E(\zeta,t).$$

Let us put $\varepsilon = \left(1+|\zeta|\right)^{-2(k+\alpha)/(2+k+\alpha)}$ form the left of inequality (3.9) we get

$$E(\zeta,t) \le \left(1+|\zeta|\right)^{2(k+\alpha)/(2+k+\alpha)} E_{\varepsilon}(\zeta,0)\exp\left(C_{2}|\zeta|^{2/(2+k+\alpha)}\right)$$

By substituting $t = 0$ into the right side of inequality (3.9), we obtain

$$E(\zeta,t) \le \left(1+|\zeta|\right)^{2(k+\alpha)/(2+k+\alpha)}(1+\Lambda)E(\zeta,0)\exp\left(C_{2}|\zeta|^{2/(2+k+\alpha)}\right) \quad (3.10)$$

when initial data g_0 and g_1 of (3.1) belong to $\mathcal{D}^{M}$ we obtain by Paley-Wiener theorem:

$$|v'(\zeta,0)|^{2}+|\zeta|^{2}|v(\zeta,0)|^{2} = E(\zeta,0) \le \left(1+|\zeta|^{2}\right)C_{0}.\exp\left(-2M\left(\frac{|\zeta|}{h}\right)\right)$$

we can rewrite (3.10) as

$$E(\zeta,t) \le \left(1+|\zeta|\right)^{2(k+\alpha)/(2+k+\alpha)}(1+\Lambda)\left(|\zeta|^{2}+1\right)C_{0}.\exp\left(-2M\left(\frac{|\zeta|}{h}\right)+C_{2}|\zeta|^{2/(2+k+\alpha)}\right)$$

by the condition (3.2) involves we obtain

$$|v'(\zeta,t)|^{2}+|\zeta|^{2}|v(\zeta,t)|^{2} = E(\zeta,t) \le \left(1+|\zeta|^{2}\right)C_{4}.\exp\left(-2M\left(|\zeta|\right)\right).$$

In the second case in which $a(t)$ to $C^{0,\alpha}([0,L])$ we take in this case

$$a_{\varepsilon} = \frac{1}{\varepsilon}\int_{0}^{+\infty}\tilde{a}(t+s)\rho\left(\frac{s}{\varepsilon}\right)ds + \varepsilon^{\alpha}I$$

where $\tilde{a}$ is the extension of a to $[0,+\infty[$, satisfying $\tilde{a} \equiv a(L)$ on $[L,+\infty[$. and ρ is non negative function in $D(]0,1[)$ such that $\int_{-\infty}^{+\infty} \rho ds = 1$. The α -continuity of a gives

$$\int_0^L \left| \left(a_\varepsilon^{'} \zeta, \zeta \right) \right| ds \le C_5 \varepsilon^{\alpha-1} |\zeta|^2$$

And

$$\int_0^L \left| \left(\left(a_\varepsilon - a \right) \zeta, \zeta \right) \right| ds \le C_6 \varepsilon^{\alpha} |\zeta|^2$$

while by definition

$$\left(a_\varepsilon \zeta, \zeta \right) \ge \varepsilon^{\alpha} |\zeta|^2$$

we can write (3.5) as

$$E_s(\zeta,t) \le E_\varepsilon(\zeta,0) \exp\left[C_6 \left(\varepsilon^{-1} + \varepsilon^{\alpha/2} |\zeta| \right) \right]$$

let us put $\varepsilon = \left(1+|\zeta|\right)^{-2/(2+\alpha)}$ form the left of inequality (3.11) we get

$$E(\zeta,t) \le (1+|\zeta|)^{\frac{2}{2+\alpha}} E_\varepsilon(\zeta,0) \exp\left[C_6 \left(|\zeta| (1+|\zeta|)^{-\frac{\alpha}{2+\alpha}} + (1+|\zeta|)^{\frac{2}{2+\alpha}} \right) \right].$$

And form the right of inequality (3.9) we have

$$E(\zeta,t) \le \left(1+|\zeta|\right)^{2/(2+\alpha)} (1+\Lambda) E(\zeta,0) \exp\left[C_7 \left(|\zeta|^{2/(2+\alpha)} \right) \right],$$

By condition (3.2) we proved the second case. We now present examples of sequences that satisfy the conditions established in the theorem

III.3 EXAMPLE

Consider the following sequences

1. $M_{1,m} = e^{rm} \Gamma(rm+1)$
2. $M_{2,m} = (m)^{rm}$

where Γ is Gamma function, and $r = 1 + \dfrac{k+\alpha}{2}$. It can be easily verified that condition (3.2) holds by applying the inequality

$$(m!)^r \le m^{rm} \le e^{rm} \Gamma(rm+1),$$

and using **Definition 2.3**. This leads to

$$M_1\left(\frac{t}{h}\right) \le M_2\left(\frac{t}{h}\right) \le t^{\frac{2}{2+k+\alpha}}, \quad \forall t \in \mathbb{R}_+^*,$$

for some $h \in \mathbb{R}^+$.

IV. COUNTER EXAMPLE

In this section, we adopt a counter example similar to the one in [4], [3] within the framework of M-ultradifferentiable classes, focusing on the initial value problem in the case where $N = 1$.

$$\begin{cases} u_{tt} = a(t) u_{xx} & \text{for} \quad x \in \mathbb{R}, \quad t \ge 0 \\ u(x,0) = g_0(x), \\ u_t(x,0) = g_1(x) \end{cases}$$

With $a \ge 0$ and g_0, g_1 are belong to $\mathcal{E}^*$. Our purpose is to construct, for any integer $(k \ge 0)$ and $0 \le \alpha < 1$, a coefficient a of class $C^{\alpha,k}$ such that the problem (4.1) becomes ill-posed in the space $\mathcal{E}^*$ if the condition (3.2) is not satisfied. We start by constructing *a* as a positive function of class $C^{\alpha,k}$. Furthermore, for every $L > 0, a$ remains nonzero on [0, L [and identically zero on $[L,+\infty[$.

Consider a positive sequence $\{\tilde{n}_n\}_{n..1}$ that decreases to zero and satisfies the condition:

$$\sum_{n=1}^{\infty} \tilde{n}_n = L, \qquad (4.2)$$

A positive sequence $\{\delta_n\}_{n..1}$ that decreases to zero, and a sequence $\{\nu_n\}_{n..1}$ of integers greater than or equal to 1, tending to infinity. Let us consider points t_n in the interval $[0, L[$ as

$$t_n = \tilde{n}_1 + \ldots + \tilde{n}_{n-1} + \frac{\tilde{n}_n}{2},$$

And the intervals

$$J_n = \left[t_n - \frac{\tilde{n}_n}{2}, t_n + \frac{\tilde{n}_n}{2} \right[.$$

We have then

$$\left[0, L \right[= \bigcup_{n=1}^{\infty} J_n .$$

Where

$$J_1 = [0, \tilde{n}_1[$$

$$J_2 = [\tilde{n}_1, \tilde{n}_1 + \tilde{n}_2[$$

$$J_3 = [\tilde{n}_1 + \tilde{n}_2, \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3[$$

$$\vdots$$

The function $a(t)$ will be defined progressively over each interval J_j, using two auxiliary functions, $\beta(\tau)$ and $\gamma(\tau)$. For $\gamma(\tau)$, we take an arbitrary C^∞ function on ¡ , which is strictly increasing on [0,1], equals zero on $]-\infty, 0]$, and 1 on $[1, +\infty[$. We take

$$\beta(\tau) = 1 - \frac{4}{10}\sin 2\tau - \frac{1}{100}(1 - \cos 2\tau)^2$$

$\beta(\tau)$ is π-periodic and valued in $\left[\frac{1}{2}, 2\right]$.

We now introduce the definition of *a* as follows:

$$\begin{cases} a = a_n b_n + a_{n-1}(1 - b_n) & \text{on } J_n \text{ for } n..1, \\ a \equiv 0 & \text{on } [L, +\infty[. \end{cases}$$

where the functions a_n and b_n are defined by:

$$\begin{cases} a_n(t) = \delta_n \cdot \beta\left(2\nu_n \pi \dfrac{t - t_n}{\rho_n}\right), n..1, \\ b_n(t) = \gamma\left(8\nu_n \dfrac{t - (t_n - \rho_n / 2)}{\rho_n}\right), n..1, \qquad (4.4) \\ a_0(t) = 2\delta_1. \end{cases}$$

We differentiate (4.3) k -times on each interval J_n we get

$$a^{(k)}\Big|_{J_n} = \sum_{l=0}^{k} \binom{k}{l} b_n^{(k-l)} \cdot \left(a_n^{(l)} - a_{n-1}^{(l)}\right) + a_{n-1}^{(k)},$$

using the Hölder norm definition (2.1) we obtain the estimate

$$\| a \|_{C^{\alpha,k}(J_n)} \leq C_{k,\alpha}\delta_{n-1}\left(\frac{\nu_n}{\rho_n}\right)^{k+\alpha}.$$

Next, we derive a condition that guarantees *a* belongs to $C^k([0,+\infty[)$, which is:

$$\delta_{n-1}\left(\frac{\nu_n}{\rho_n}\right)^{k} \to 0 \quad \text{as } n \to \infty.$$

We can find a sufficient condition for *a* to belong to $C^{\alpha,k}([0,+\infty[)$ as:

$$\delta_{n-1}\left(\frac{\nu_n}{\rho_n}\right)^{k+\alpha} \leq M, \quad \forall_n.$$

We introduce the notation:

$$\begin{cases} t'_n = \left(t_n - \dfrac{\rho_n}{2}\right) + \dfrac{\rho_n}{8\nu_n}, \\ t''_n = \left(t_n + \dfrac{\rho_n}{2}\right) - \dfrac{\rho_n}{8\nu_n}. \end{cases}$$

We divide the intervals J_n at t'_n into

$$I_n = [t'_n, t_n + \frac{\rho_n}{2}[\quad \text{and} \quad \bar{I}_n = [t_n - \frac{\rho_n}{2}, t'_n[$$

We define the solution *u* as follows:

$$u(x,t) = \sum_{n=1}^{\infty} v_n(t)\sin\left(h_n x\right)$$

with

$$h_n = 2\pi\frac{\nu_n}{\rho_n}\frac{1}{\sqrt{\delta_n}}$$

and v_n is the solution of the following system:

$$\begin{cases} v'' = -h_n^2 a(t) v, \\ v(t_n) = 0, \\ v'(t_n) = 1. \end{cases}$$

As $a(t) \equiv \delta_n \cdot \beta\left(2\pi\left(\nu_n/\rho_n\right)\left(t - t_n\right)\right)$ on I_n, due to $b_n(t) = \gamma\left(8\nu_n\dfrac{t - (t_n - \rho_n/2)}{\rho_n}\right) = 1$ on I_n, we can determine v_n on I_n as

$$v_n(t) = \frac{\rho_n}{2\pi\nu_n} w\left(2\pi\frac{\nu_n}{\rho_n}(t - t_n)\right), \quad \text{on } I_n,$$

where w solution of

$$\begin{cases} w'' + \beta(\tau) w = 0, \quad \text{on } \mathbb{R} \\ w(0) = 0, \quad w'(0) = 1, \end{cases}$$

the solution of (4.10) is given by

$$w(\tau) = \sin(\tau)\cdot\exp\left[\frac{1}{10}\left(\tau - \frac{1}{2}\sin 2\tau\right)\right]$$

If we define the energy corresponding to the function v_n as

$$E_n(t) = h_n^2 a(t) v_n^2 + v_n'^2,$$

As a consequence, (4.9) and (4.11) provide an explicit representation of v_n on I_n, and specifically,

$$\begin{cases} |v_n(t_n')| = \tilde{C}_1 \dfrac{\tilde{n}_n}{v_n} \exp\left(-\dfrac{\pi}{10} v_n\right) \\ |v_n'(t_n')| = \tilde{C}_2 \exp\left(-\dfrac{\pi}{10} v_n\right) \end{cases}.$$

Then we substitute (4.13) into (4.12), and we get

$$E_n(t_n') = C_0 \exp\left(-\frac{\pi}{5} v_n\right).$$

We differentiate (4.12) and integrate from *t* to *s* obtaining the following estimate:

$$\begin{aligned} E_n(s) &\le E_n(t) + \int_t^s \frac{|a'(\eta)|}{a(\eta)} (E_n(\eta) - v_n'^2(\eta)) d\eta \\ &\le E_n(t) + \int_t^s \frac{|a'(\eta)|}{a(\eta)} E_n(\eta) d\eta \quad t < s \end{aligned}.$$

By applying the Gronwall lemma, we obtain

$$E_n(s) \le E_n(t) \exp\left[\int_t^s \frac{|a'(\eta)|}{a(\eta)} d\eta\right], \quad t < s$$

We apply (4.15) with $t = t_n'$ and $t_n' < s$. Therefore, we need to estimate the integral

$$\int_{t_n'}^s \frac{|a'|}{a} d\eta.$$

To summarize the behavior of a, we consider its properties over the interval:

$$[0, t_n'] = \bar{I}_1 \cup I_1 \cup \ldots \cup \bar{I}_{n-1} \cup I_{n-1} \cup \bar{I}_n,$$

as given below:

1. The function *a* decrease near $t = 0$ and $t = t_n'$.
2. The values of *a* at specific points are given by $a(0) = 2\delta_1$ and $a(t_n') = c_1 \cdot \delta_1$, where $c_1 = \beta(\pi/4)$.
3. The function *a* attains exactly $2v_k$ local minima and $2v_k$ local maxima within I_k, satisfying the bound

$$\frac{\delta_k}{2} \le a(t) \le 2\delta_k.$$

4. The function *a* is decreasing in a neighborhood of $\bar{I}_k$. To satisfy the condition (4.16) we require a further assumption on the parameters, namely that

$$2\delta_n \le \frac{\delta_{n-1}}{2}, \quad \forall n.$$

By utilizing the properties of a described above, we obtain from (4.15).

$$E_n(s) \le E_n(t_n') \exp\left[2(v_1 + \ldots + v_{n-1}) \lg 4 + \lg\left(\frac{2}{c_1} \cdot \frac{\delta_1}{\delta_n}\right)\right].$$

Alternatively, the Paley-Wiener theorem for M-ultradifferentiable classes guarantees that the series (4.7) converges to the function $u(x,t)$ in $C([0, L-\varepsilon], E^*)$ for some $\varepsilon > 0$ and a certain $h \in \mathbb{R}^+$ (or for any $h \in \mathbb{R}^+$), if and only if

$$\sup_{[0,L-\varepsilon]} |v_n| \leq C_{h,\varepsilon}\cdot\exp(-M(h_n/h)).$$

Where $C_{h,\varepsilon}\in\mathbb{R}^{+}$. Therefore, since $t_{n'}\to L$ as $n\to\infty$, we derive from (4.18) the following sufficient condition for $u(x,t)$ to be included in $C([0,L[,\mathcal{E}^{*})$:

$$M(h_n/h) \leq \frac{\pi}{10}\nu_n-(\nu_1+...+\nu_{n-1})\lg 4-\lg\frac{\delta_1}{\delta_n}+\lg C_h.$$

Where $C_h\in\mathbb{R}^{+}$. Taking into account that $h_n=2\pi\nu_n/\rho_n\sqrt{\delta_n}$,we find that (4.19) holds when

$$(\nu_1+\dots+\nu_{n-1})\log 4 \leq \frac{\pi}{10}\nu_n,$$

And

$$\sup_{p\in\mathbb{N}}\left(\log\frac{h_n^{p}}{hM_p\delta_n}\right)<c_h.$$

Where ch constant depend with h and δ_1. The successful construction of the function $a(t)$ enabled us to obtain a solution u in the form of a convergent series in $C^{2}\left([0,L],\mathcal{E}^{*}\right)$. Moreover, we observe from (4.9) and (4.11) that

$$\begin{cases}\left|v_n\left(t_n''\right)\right|=\tilde{\tilde{C}}\dfrac{\rho_n}{\nu_n}\exp\left(\dfrac{\pi}{10}\nu_n\right)\\ \left|v_n'\left(t_n''\right)\right|=\tilde{C}\exp\left(\dfrac{\pi}{10}\nu_n\right)\end{cases},$$

with $\tilde{\tilde{C}}$ and $\tilde{C}$ are positive constants. After substituting h_n we obtained

$$\left|v_n\left(t_n''\right)\right| \geq \frac{1}{\tilde{\tilde{\tilde{C}}}}\exp\left(\frac{\mu}{2}h_n\right),$$

where $\tilde{\tilde{\tilde{C}}}$ and μ are positive constants. Inequality (4.23) shows that $\{u(\cdot,t_n'')\}$ is unbounded in the space of distributions. The sequences ρ_n, $\mathbf{v}_n$, and δ_n are chosen as in [4], satisfying the conditions (4.2), (4.5),(4.6), (4.17), (4.20) and (4.21).

$$\begin{cases}\rho_n=2^{-n}T_*\\ \nu_n=2^{n^2}\\ \delta_n=2^{-(k+\alpha)(n+1)(n+2)-2n}\end{cases}$$

In particular

$$h_n=\frac{2\pi}{L}2^{n^2+2n+(k+\alpha)(n+1)(n+2)/2}.$$

V. CONCLUSION AND FUTURE WORK

In conclusion, this paper addresses the weakly hyperbolic Cauchy problem by considering Hölder's regularity of coefficients dependent on time within the framework of M-ultradifferentiable well-posedness. By formulating an equivalent condition of well-posedness within the framework of Gevrey regularity, we guarantee the well-posedness of the problem for a family of M-ultradifferentiable functions, characterized by the function corresponding to the sequence *M*. In the future, this condition can be used as a starting point to derive conditions ensuring the stability and solvability of weakly hyperbolic Cauchy problems within ultradistributions of classes *M*. This research opens up new possibilities for investigating the relationship between the identified condition and the solvability of such problems in the framework of ultradistributions. The findings have the potential to contribute to the development of analytical techniques and criteria for studying and applying ultradistributions in various mathematical models.

VI. AUTHOR'S CONTRIBUTION

Conceptualization: All authors.
Methodology: All authors.
Investigation: All authors.
Discussion of results: All authors.

Writing – Original Draft: Author One, Author Two.
Writing – Review and Editing: All authors.
Resources: All authors.
Supervision: All authors.
Approval of the final text: Author One, Author Two and Author Three.

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