



## RESEARCH ARTICLE

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# NUMERICAL SOLUTIONS FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL CAPUTO-HADAMARD OPERATOR WITH SIMULATION ON PHYSICS

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## ABSTRACT

This paper presents a novel contribution focused on constructing numerical solutions for a stochastic fractional differential equation governed by two distinct operators: the Caputo-Hadamard and the Caputo-Katugampola fractional derivatives. These operators enable the evaluation and validation of the most accurate approximations for the given problem. Building on these results, we extend our approach to solve fractional neutron point kinetic equations, which model anomalous slow diffusion in nuclear reactor dynamics. Numerical techniques, such as the explicit finite difference method, are employed to efficiently and effectively derive solutions. These solutions are analyzed and graphically represented to demonstrate their accuracy and applicability. The fractional neutron point kinetic model introduces a fractional-order exponent to the relaxation time associated with neutron flux variation, providing a more refined understanding of neutron dynamics. This model is used to analyze the dynamic behavior of neutron diffusion under subcritical, supercritical, and critical reactivities. The results are compared with those obtained from classical neutron point kinetic equations, highlighting the advantages of the fractional approach in capturing complex neutron behavior. This work contributes to the broader field of nuclear reactor dynamics by offering a robust framework for modeling and simulating anomalous diffusion processes.



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## I. INTRODUCTION

Data is increasingly recognized as a vital resource in modeling, with its diverse forms originating from numerous natural sources. Physics and economics are among the fields that require vast amounts of data to advance their understanding and predictive capabilities. In physics, data is crucial for validating theories and models that describe complex phenomena, such as particle interactions and cosmological events. The precision and accuracy of these models depend heavily on the quality and quantity of data available. In economics, data plays a pivotal role in analyzing market trends, predicting economic downturns, and informing policy decisions [1], [2] and [3]. The availability of extensive datasets allows economists to develop more sophisticated models that capture the intricacies of economic systems. Moreover, advancements in data collection and analysis technologies have enabled researchers to explore new dimensions of economic behavior, leading to more nuanced insights into global economic dynamics. The integration of data from various fields also fosters interdisciplinary research, where insights from one domain can inform and enhance understanding in another. This cross-pollination of ideas and methodologies can lead to groundbreaking discoveries and innovative solutions, underscoring the importance of data in driving scientific progress across multiple disciplines [4].

In stochastic modeling, data values are analyzed through probabilistic frameworks that separate systematic trends from random fluctuations, enabling robust predictions in uncertain environments. These models prioritize parsimony, balancing parameter complexity to avoid overfitting while maintaining predictive stability a critical consideration in finance and machine learning for handling volatility and dynamic systems. Fractional models see [5-7], conversely, specialize in variables constrained to intervals such as, which are common in economics (e.g., market participation rates [1]) and environmental science (e.g., pollution indices). By leveraging link functions akin to logistic regression, fractional models accommodate continuous proportional outcomes, outperforming classical approaches in scenarios like freelancing trend predictions post-COVID. Recent advancements integrate fractional calculus with machine learning to quantify variability in big data, enhancing anomaly detection and system adaptability through physics-informed frameworks. For instance, hybrid architectures combining fractional derivatives and neural networks demonstrate improved accuracy in modeling nonlinear phenomena, such as economic shifts or biological processes. In stochastic contexts, data's value lies in its capacity to quantify uncertainty, while fractional approaches extract insights from bounded, proportional relationships-both essential for advancing predictive analytics in interdisciplinary research. Recent advances in mathematical modeling have catalyzed a critical interdisciplinary axis within the fractional differential equations research community [8], marked by the convergence of expertise in fractional calculus and stochastic processes [9]. Fractional order operators provide nuanced descriptions of power-law dynamics and long-range dependencies in systems such as anomalous diffusion or viscoelastic materials. Stochastic frameworks, meanwhile, quantify uncertainty arising from environmental noise, measurement errors, and probabilistic interactions [10]. By integrating these paradigms, researchers develop hybrid methodologies that resolve limitations in modeling multi-scale processes from turbulent fluid flows to financial markets exhibiting memory-driven volatility. For instance, fractional stochastic differential equations now underpin improved predictions of groundwater contamination patterns by coupling sub diffusive transport with random hydraulic conductivity fields. Similarly, fractional Brownian motion models enhance risk assessment in economics by capturing persistent or anti-persistent trends in asset prices. This cross-disciplinary synergy not only advances analytical and numerical techniques but also yields novel computational tools for inverse problems and parameter estimation, demonstrating how blending fractional and stochastic perspectives unlocks deeper insights into real-world systems governed by intertwined deterministic and random behaviors.

Several research papers have explored a class of stochastic models driven by operators involving fractional derivatives as highlighted in references [11] and [12]. There are papers that open the eyes of scientists to this field but not enough. Research in fractional stochastic differential equations (FSDEs) is dense due to their unique ability to model systems with memory, randomness, and complex dynamics. And provide a powerful framework for accurately describing real-world phenomena. The interdisciplinary nature of FSDEs attracts researchers from diverse fields, fostering collaboration and innovation. However, solving FSDEs is computationally demanding, driving the development of advanced numerical methods and algorithms. Additionally, the field is rich with open problems, such as extending FSDEs to high-dimensional systems and establishing rigorous mathematical foundations. As FSDEs find new applications in AI, machine learning, and quantum computing, their potential to solve complex, real-world problems continues to grow, making this a vibrant and rapidly evolving area of research. Our vision is rooted in advancing theoretical research with a strong focus on practical applications, particularly in the realm of quantum physics. By extending the theoretical foundations of fractional calculus and stochastic modeling, we aim to develop innovative frameworks that address complex quantum phenomena. These advancements will not only deepen our understanding of quantum systems but also pave the way for transformative applications in quantum computing, material science, and beyond. Through this dual emphasis on theory and application, we strive to bridge the gap between abstract mathematical concepts and real-world quantum challenges, driving progress in both fundamental science and cutting-edge technology. Our paper is structured as follows: The second section provides a comprehensive review of the prominent definitions of fractional derivative operators and stochastic calculus, illustrated through methods and approaches related to these two tools. The third section focuses on constructing numerical solutions for the proposed sampling of fractional differential equations driven by Caputo-Hadamard derivative. The fourth section is dedicated to simulations aimed at testing the influence of these operators on our proposed system. The fifth section is centered on applying the framework to a problem in quantum physics.

## II. PRELIMINARY MATHEMATICAL ON STOCHASTIC AND FRACTIONAL CALCULUS

Here we present a background about fractional and stochastic calculus for use in our physics paper

### II.1 STOCHASTIC DIFFERENTIAL EQUATIONS TOOL

Brownian motion is a fundamental concept in physics and mathematics, providing a cornerstone for understanding diffusion processes, stochastic processes, and the kinetic theory of matter. Albert Einstein, in his seminal 1905 paper, provided a theoretical explanation for Brownian motion, linking it to the existence of atoms and molecules and offering strong evidence for the atomic theory of matter. Mathematically, Brownian motion is modeled as a continuous-time stochastic process with independent, normally distributed increments, making it a key example of a Wiener process. It has wide-ranging applications, from modeling stock prices in finance to describing particle diffusion in biology and physics. Brownian motion remains a vital tool for exploring randomness and complexity in natural and engineered systems.

**Definition 1** [13] *The process  $(W(t), t \geq 0)$  is defined as Brownian motion if and only if*

- 1)  $W(0) = 0$ .
- 2)  $W(t)$  is almost continuous in  $t$ .
- 3) For all  $t > t_0$ :  $W(t) - W(t_0)$  follow the normal distribution with mean zero and variance  $(t - t_0)$ , This condition can be summarized by the following approximation

$$W(t) - W(t_0) \approx \sqrt{t - t_0} \varepsilon(t). \quad (1)$$

Where  $\varepsilon(t)$  follows a Gaussian distribution with mean 0 and variance 1.

- 4) The increments of  $W(t)$  are independent; in other words, for each order  $t_0 < t_1 < t_2 < \dots < t_n$ . Then, following differences  $W(t_1) - W(t_0)$ ,  $W(t_2) - W(t_1)$ , ...,  $W(t_n) - W(t_{n-1})$  are independent.

This section provides an overview of key concepts in fractional and stochastic calculus, focusing on the foundational tools required to analyze and solve the system under consideration. Here, we revisit essential definitions and principles that will serve as the basis for understanding the methodologies applied in this study.

$$X(t) = a_0(X(t), t)dt + a_1(X(t), t)dW(t). \quad (2)$$

Where  $a_0$  and  $a_1$  are tow mesurable functions on the interval  $[t_0; M]$ ,  $W(t)$  is Brownian motion. Then, among the fundamental theorems that ensure the existence and uniqueness of the solution of the stochastic model (2), the following theorem:

**Theorem 1** Let  $K_0$  and  $K_1$  be two positive constants, if the process (2) satisfies the following two properties

- 1) Lipshitzian condition

$$\begin{aligned} & |a_0(X_i(t), t) - a_0(X_j(t), t)| + |a_1(X_i(t), t) - a_1(X_j(t), t)| \\ & \leq K_0 |X_i(t) - X_j(t)|. \quad (3) \end{aligned}$$

- 2) Restriction on growth

$$|a_0(X_i(t), t)|^2 + |a_1(X_i(t), t)|^2 \leq K_1(1 + X^2(t)). \quad (4)$$

Then, under these conditions the process described in the process (2) admits a unique solution see the theorem with its proof in reference [4].

**Theorem 2** Suppose  $h : [0, T] \rightarrow \mathbb{R}$  is continuously differentiable under condition  $h(0) = h(T)$ . Then

$$\int_0^T h(t) dW(t) = \int_0^T h'(t) dW(t). \quad (5)$$

A stochastic differential equation can be discretized into a nonlinear time series, specifically the stochastic differential equation known in the stochastic literature as the continuous bilinear stochastic equation, see [14] and [15]. Indeed, Lawrence C. Evans in [13] and other researchers in stochastic analysis have demonstrated that the derivative of a Brownian motion "Wiener process" can be interpreted as white noise.

$$\frac{dW(t)}{dt} = \varepsilon(t). \quad (6)$$

This result is fundamental in the study of stochastic processes and has significant theoretical and practical implications. Interpreting  $\frac{dW(t)}{dt}$  as white noise is crucial for modeling systems influenced by random fluctuations. Examples include thermal noise in electrical circuits (physics), random shocks in asset prices in finance see [13], signal processing, and control systems engineering's Equation (6)

**Definition 2** Consider a first-order continuous-time bilinear process "Black-Scholes Model" is governed by the following SDE

$$\begin{cases} dX(t) = (\alpha(t)X(t) + \alpha_0(t))dt + (\beta(t)X(t) + \beta_0(t))dW(t) \\ X(t_0) = X(0) \end{cases}. \quad (7)$$

In which  $(\alpha(t), \beta(t), \alpha_0(t), \beta_0(t))$  are measurable deterministic functions to the conditions,  $\forall t \geq 0$ ,  $\alpha(t) \neq 0$ , and  $\alpha(t)\beta_0(t) \neq \beta(t)\alpha_0(t)$ . The existence and the uniqueness of the solution process  $(X(t))_{t \geq 0}$  of SDE (1.1) are ensured by general results on stochastic differential equations (see [16]). In order to ensure the stationarity of the solution process, we require that the parameters  $\alpha$  and  $\beta$  are subject to the following condition (see Le Breton and Musiela m 1984 in [15]. And in another way, we keep the following bounding conditions, for all  $t$  we have

$$\int_0^T |G(t)|^i dt < \infty, \quad i = 1, 2. \quad (8)$$

Where  $G(t) = \alpha(t), \beta(t), \alpha_0(t)$  or  $\beta_0(t)$ . The existence and uniqueness of Ito solution process  $(X(t), t \geq 0)$  defined by its general stochastic bilinear equation (7) under the above assumptions see [14] accepts solution formulated by the following expression

$$X(t) = \Phi(t) \left[ X(0) + \int_0^t \beta_0(S) \Phi^{-1}(S) (\alpha_0(S) - \alpha(S)) dS + \int_0^t \Phi^{-1}(S) \beta_0(S) dB(S) \right]. \quad (9)$$

$$\text{Where } \Phi(t) = \exp \left\{ \int_0^t \left( \alpha(S) - \frac{1}{2} \beta^2(S) \right) dS + \int_0^t \beta(S) dB(S) \right\}. \quad (10)$$

Several applications in economics and finance the derivation of  $dB(S)$  takes a COGARCH model see references [3] and [17]. Fractional differentiation extends derivatives to non-integer or complex orders, offering greater flexibility than integer-order derivatives. Originating from Leibniz and Liouville, it is widely applied in physics, engineering, and applied sciences. Key definitions include Riemann-Liouville, Caputo, and Grünwald-Letnikov, each suited for modeling complex phenomena like long-memory systems or anomalous diffusion. The Euler-Maruyama method is the simplest and most straightforward time-discrete approximation for simulating Itô processes, its expression discretization as follows

$$dX(t_n) = a_0(X(t_n), t_n) dt_n + a_1(X(t_n), t_n) dW(t_n). \quad (11)$$

For given time discretization

$$\Delta X(t_n) = a_0(X(t_n), t_n) \Delta t_n + a_1(X(t_n), t_n) \Delta W(t_n). \quad (12)$$

Where  $\Delta X(t_n) = X(t_{n+1}) - X(t_n)$ ,  $\Delta t_n = t_{n+1} - t_n$  and  $\Delta W(t_n) = W(t_{n+1}) - W(t_n)$ . Here each random number  $\Delta W(t_n)$  is computed as

$$\Delta W(t_n) = \varepsilon(n) \sqrt{\Delta t_n}. \quad (13)$$

Where  $\varepsilon(n)$  is chosen from the standard normal distribution.

**Definition 3** [18] Let be  $[a, b] (-\infty < a < b < +\infty)$  a finite interval of  $\mathbb{R}$ , the left and right Riemann-Liouville fractional integrals  $R_{(a,t)}^\gamma$  and  $R_{(t,b)}^\gamma$  of order  $\gamma$  are defined as follows

$$R_{(a,t)}^\gamma(X(t)) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-S)^{\gamma-1} X(S) dS, \quad t > a, \quad \gamma > 0. \quad (14)$$

And

$$R_{(t,b)}^\gamma(X(t)) = \frac{1}{\Gamma(\gamma)} \int_t^b (S-t)^{\gamma-1} X(S) dS, \quad t < b, \quad \gamma > 0. \quad (15)$$

**Definition 4** Where the gamma function defined by its formula

$$\Gamma(\gamma) = \int_0^{+\infty} S^{\gamma-1} \exp(-S) dS. \quad (16)$$

**Definition 5** [19] The left and right Riemann-Liouville (RL) fractional derivatives  $D_{(a,t)}^{RL(\gamma)}$  and  $D_{(t,b)}^{RL(\gamma)}$  of order  $\gamma > 0$ , are defined as follows

$$D_{(a,t)}^{RL(\gamma)} X(t) = \frac{d^N}{dt^N} R_{(a,t)}^{N-\gamma}(X(t)). \quad (17)$$

Then

$$D_{(a,t)}^{RL(\gamma)} X(t) = \frac{1}{\Gamma(N-\gamma)} \frac{d^N}{dt^N} \int_a^t (t-S)^{N-\gamma-1} X(S) dS. \quad (18)$$

Similarly, the right-sided Riemann-Liouville fractional derivative of order  $\gamma$  is given by

$$D_{(t,b)}^{RL(\gamma)} X(t) = (-1)^N \frac{d^N}{dt^N} (R_{(t,b)}^{N-\gamma} X(t)), \quad (19)$$

which can also be written in integral form as

$$D_{(t,b)}^{RL(\gamma)} X(t) = (-1)^N \frac{1}{\Gamma(N-\gamma)} \frac{d^N}{dt^N} \int_t^b (S-t)^{N-\gamma-1} X(S) dS. \quad (20)$$

Where  $N = [\gamma] + 1$ , such as  $[\gamma]$  denotes the smallest integer greater than or equal to  $\gamma$ .

**Definition 6** [20] and [18] The left and right Caputo fractional derivative  $D_{(a,t)}^{C(\gamma)}X(t)$  and  $D_{(t,b)}^{C(\gamma)}X(t)$  of order  $\gamma > 0$  are defined by

$$D_{(a,t)}^{C(\gamma)}X(t) = D_{(a,t)}^{RL(\gamma)} \left\{ X(t) - \sum_{j=0}^{N-1} \frac{X^{(j)}(a)}{j!} (t-a)^j \right\}. \quad (21)$$

The formula can be extended as follows

$$D_{(a,t)}^{C(\gamma)}X(t) = D_{(a,t)}^{RL(\gamma)}X(t) - D_{(a,t)}^{RL(\gamma)} \left\{ \sum_{j=0}^{n-1} \frac{X^{(j)}(a)}{j!} (t-a)^j \right\}. \quad (22)$$

Then

$$D_{(a,t)}^{C(\gamma)}X(t) = D_{(a,t)}^{\gamma}X(t) - \sum_{j=0}^{N-1} \frac{X^{(j)}(a)}{\Gamma(j-\beta+1)} (t-a)^{j-\gamma}, \quad (23)$$

The same principle applies too  $D_{(t,b)}^{C(\gamma)}X(t)$

$$D_{(t,b)}^{C(\gamma)}X(t) = D_{(t,b)}^{\gamma}X(t) - \sum_{j=0}^{N-1} \frac{X^{(j)}(b)}{\Gamma(j-\beta+1)} (b-t)^{j-\gamma}. \quad (24)$$

Where  $N = [\gamma] + 1$ , for  $0 < \gamma < 1$  we obtain

$$D_{(a,t)}^{C(\gamma)}X(t) = D_{(a,t)}^{\gamma}X(t) - \frac{X(a)}{\Gamma(1-\gamma)} (t-a)^{-\gamma}. \quad (25)$$

The same situation for  $D_{(t,b)}^{C(\gamma)}X(t)$  we find that

$$D_{(t,b)}^{C(\gamma)}X(t) = D_{(t,b)}^{\gamma}X(t) - \frac{X(b)}{\Gamma(1-\gamma)} (b-t)^{-\gamma}. \quad (26)$$

**Definition 7** [21] The left side Hadamard fractional derivative of order  $\gamma > 0$  of continuous function  $X(t)$  on interval  $(a,b)$  is given by

$$D_{(a,t)}^{H(\gamma)}X(t) = \frac{1}{\Gamma(N-\gamma)} \left( t \frac{d}{dt} \right) \int_a^t \frac{1}{S} \log \left( \frac{t}{S} \right)^{N-\gamma-1} X(S) dS. \quad (27)$$

**Definition 8** Caputo Hadamard formulation derivative given by the following expression

$$D_{(a,t)}^{H(\gamma)}X(t) = \frac{1}{\Gamma(N-\gamma)} \left( t \frac{d}{dt} \right) \int_a^t \frac{1}{S} \log \left( \frac{t}{S} \right)^{N-\gamma-1} X(S) dS. \quad (28)$$

In case where  $0 < \gamma < 1$ , we come for the formula

$$D_{(a,t)}^{H(\gamma)}X(t) = \frac{1}{\Gamma(1-\gamma)} \left( t \frac{d}{dt} \right) \int_a^t \frac{1}{S} \log \left( \frac{t}{S} \right)^{-\gamma} X(S) dS.$$

### III. NUMERICAL DISCRETIZATION TECHNIQUES FOR CAPUTO-HADAMARD FRACTIONAL OPERATOR

Numerical discretization techniques for the Caputo-Hadamard fractional operator play a crucial role in solving fractional differential equations that arise in various scientific and engineering applications. The Caputo-Hadamard operator, which combines the features of Caputo-type fractional derivatives with the logarithmic kernel of Hadamard integrals, presents unique challenges due to its non-local and singular nature. To address these challenges, researchers have developed advanced numerical methods, such as finite difference schemes, spectral methods, and quadrature-based approaches, to approximate the operator efficiently. These techniques often involve careful handling of the logarithmic singularity and adaptive strategies to ensure accuracy and stability. Recent advancements have also explored high-order discretization methods and hybrid algorithms to improve computational efficiency for complex systems. By leveraging these numerical techniques, the Caputo-Hadamard fractional operator can be effectively applied to model real-world phenomena, such as anomalous diffusion, viscoelasticity, and processes with memory effects, providing deeper insights into their underlying dynamics. Now, we are applying our discretization using the Caputo-Hadamard method for discretization, let the points  $a = t_0 < t_1 < \dots < t_N$ . Such as  $a = t_0 < t_1 < \dots < t_N$ . Such as  $t_{i+1} - t_i = l$

$$\begin{cases} D_{(a,t)}^{CH(\gamma)} X(t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t \log\left(\frac{t}{s}\right)^{-\gamma} \delta X(s) \frac{ds}{s} \cdot (29) \\ 0 < \gamma < 1 \end{cases}$$

$$\begin{aligned} D_{(a,t)}^{CH(\gamma)} X(t) &\approx \frac{1}{\Gamma(1-\gamma)} \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \log\left(\frac{t}{s}\right)^{-\gamma} \delta X(s) \frac{ds}{s}, \delta = \left(s \frac{d}{ds}\right) \\ &= \frac{1}{\Gamma(1-\gamma)} \sum_{n=1}^N \int_{t_{j-1}}^{t_j} t_j \frac{X(t_{j+1}) - X(t_j)}{t_{j+1} - t_j} \log\left(\frac{t}{s}\right)^{-\gamma} \\ &\quad \times \delta X(s) \frac{ds}{s} \\ &= \frac{1}{l \cdot \Gamma(1-\gamma)} \sum_{j=0}^N t_j (X(t_{j+1}) - X(t_j)) \\ &\quad \times \int_{t_j}^{t_{j+1}} \log\left(\frac{t}{s}\right)^{-\gamma} \frac{ds}{s}. \end{aligned}$$

Then

$$\begin{aligned} D_{(a,t)}^{CH(\gamma)} X(t) &= \frac{1}{l \cdot \Gamma(1-\gamma)} \sum_{j=0}^N t_j (X(t_{j+1}) - X(t_j)) \\ &\quad \times \left[ \left( \log \frac{t_{n+1}}{t_j} \right)^{1-\gamma} - \left( \log \frac{t_{n+1}}{t_{j-1}} \right)^{1-\gamma} \right]. (30) \end{aligned}$$

Using this discretization, the Caputo-Katugampola operator  $D_{(T_0,t)}^{CK(\gamma,\theta)} X(t_s)$  can be approximated as:

$$\begin{aligned} D_{(T_0,t)}^{CK(\gamma,\theta)} X(t_s) &= \frac{\theta^\gamma}{\Gamma(2-\gamma)} \sum_{j=1}^N \frac{X(j) - X(j-1)}{t_j - t_{j-1}} t_j^{1-\gamma} \\ &\quad \times \left[ \left( t_N^\theta - t_j^\theta \right)^{1-\gamma} - \left( t_N^\theta - t_{j-1}^\theta \right)^{1-\gamma} \right]. (31) \end{aligned}$$

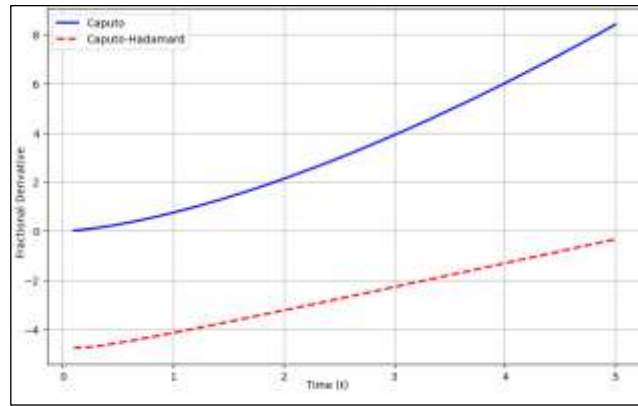


Figure 1: Comparison of Caputo and Caputo-Hadamard Derivatives  $\alpha=0.5$ , function  $X(t)=2t$  with subdivision  $N=100$ .  
Source: Authors, (2026).

#### IV. ILLUSTRATION AND APPLICATION ON PHYSICS

The diffusion theory model of neutron transport is a fundamental tool in reactor theory, balancing simplicity and realism for preliminary design and scientific analysis. It simplifies neutron behavior by characterizing neutrons with a single energy or speed, making it essential for studying nuclear engineering problems. Derived from Fick's law, the neutron diffusion equation underpins this model, though it relies on simplifying assumptions, leading to the diffusion approximation. While advanced transport codes are used for complex scenarios like shielding calculations, simpler models like two-group diffusion theory remain valuable for understanding neutron dynamics. In thermal reactors, neutrons are often divided into thermal (slow) and fast groups, each described by group constants such as diffusion coefficients and cross sections, which capture interactions like scattering and absorption. In the classical laws of radioactivity, the rate of change of the number of original nuclei with respect to time is described by the following expression

$$-\frac{dX(t)}{dt} = C_0 X(t). \quad (32)$$

In critical scenarios, governing equations often become complex, yielding explicit solutions for parameters like flux distribution.

Fractional differential equations are vital for modeling anomalous neutron diffusion, with fractional neutron point kinetic equations providing deep insights into reactor dynamics under varying reactivity. To address random fluctuations in neutron density and precursor concentrations, especially at low power levels, stochastic methods like Euler-Maruyama and Taylor techniques are employed. Numerical approaches, such as the differential transform method (DTM) and its multistep variant (MDTM), efficiently solve classical point kinetic equations, while the explicit finite difference method is used for fractional stochastic models. These advancements enhance the analysis of neutron behavior under fractional-order conditions, improving reactor safety and performance understanding. By combining analytical, numerical, and stochastic methods, researchers can better predict and optimize reactor dynamics, ensuring safe and efficient operation while advancing nuclear engineering. A stochastic fractional differential equation (SFDE) modeling neutron behavior could capture randomness or uncertainty in neutron transport, diffusion, or other processes. Fractional calculus effectively describes anomalous diffusion or memory effects, while stochastic terms account for system noise or randomness. Below is an example of an SFDE that might describe neutron behavior in a medium.

$$\begin{cases} dX(t) = C_0(t)X(t)dt + C_1(t)X(t)dW(t) \\ \quad + C_2(t)X(t)D_{(T_0,t)}^{CH(\gamma)}W(t). \\ 0 < \gamma \leq 1. \end{cases} \quad (33)$$

$X(t)$ : The neutron density or number of neutrons at time  $t$ ,  $dX(t)$  the fractional derivative with respect to time. This represents memory effects or non-local behaviour in time.  $C_0(t)$ ,  $i = 1, 2, 3$ : The diffusion coefficient is a fundamental parameter in neutron transport theory that quantifies the rate at which neutrons spread or diffuse through a medium. It governs the spatial distribution of neutrons and depends on the material properties of the medium, such as the scattering and absorption cross-sections of the nuclei. introduces randomness or noise, which could represent uncertainties in neutron interactions, external influences, or measurement errors. First we approximate the following fractional stochastic term  $D_{(T_0,t)}^{CH(\gamma)}W(t)$  as follows using Caputo Hadamard discretization, using property (13) we will find that

$$D_{(T_0,t)}^{CH(\gamma)}W(t) = \frac{1}{\Gamma(1-\gamma)} \sum_{j=0}^N t_j (W(t_{j+1}) - W(t_j)) \times \left[ \left( \log \frac{t_{n+1}}{t_j} \right)^{1-\gamma} - \left( \log \frac{t_{n+1}}{t_{j-1}} \right)^{1-\gamma} \right]$$

The discretization of our proposed equation (33) is

$$D_{(T_0,t)}^{CH(\gamma)}W(t) = \frac{1}{\Gamma(1-\gamma)} \sum_{j=0}^N t_j \left( \varepsilon(n) \sqrt{\Delta t_j} \right) \times \left[ \left( \log \frac{t_{n+1}}{t_j} \right)^{1-\gamma} - \left( \log \frac{t_{n+1}}{t_{j-1}} \right)^{1-\gamma} \right]. \quad (34)$$

We discuss the solution of our system in the case  $t_j \rightarrow t_{j+1}$ , which shows that the fractional derivation term in the sense of Caputo Hadamard of  $D_{(T_0,t)}^{CH(\gamma)}W(t)$  tends towards zero. Then, the system becomes purely stochastic:

$$dX(t) = C_0(t)X(t)dt + C_1(t)X(t)dW(t) + C_3(t)dW(t). \quad (35)$$

We can write the formula in the form:

$$dX(t) = C_0(t)X(t)dt + [C_1(t)X(t) + C_3(t)]dW(t).$$

And it accepts an exact solution in the sense of **Itô** under the following form (9), where:

$$X(t) = \beta(t) \left[ X(0) - \int_0^t \beta^{-1}(s) (C_1(s) C_3(s)) ds + \int_0^t \beta^{-1}(s) C_3(s) dW(s) \right]. \quad (36)$$

Where:

$$\beta(t) = \exp \left[ \int_0^t \left( C_0(s) - \frac{1}{2} C_1^2(s) \right) ds + \int_0^t C_1(s) dW(s) \right]. \quad (37)$$

## V. SIMULATION

Let the following equation with constant coefficients

$$\begin{cases} dX(t) = 0.5X(t)dt + 0.2X(t)dW(t) \\ \quad + \frac{3}{2}X(t)D_{(0,t)}^{CH(\gamma)}W(t) + C_3(t)dW(t) \\ X(0) = 3, \quad W(t) = \sqrt{0.2}\dot{W}(t), \quad N = 100. \end{cases} \quad (38)$$

Table 1: Comparing Stochastic and Fractional Stochastic Solution.

| Fractional order | Stochastic solution | Numerical fractional Solution |
|------------------|---------------------|-------------------------------|
| 0.1              | 3.28                | 3.1123                        |
| 0.2              |                     | 3.1345                        |
| 0.3              |                     | 3.2456                        |
| 0.4              |                     | 3.3451                        |
| 0.5              |                     | 3.1999                        |
| 0.6              |                     | 3.2766                        |
| 0.7              |                     | 3.2788                        |
| 0.8              |                     | 3.2805                        |

Source: Authors, (2026).

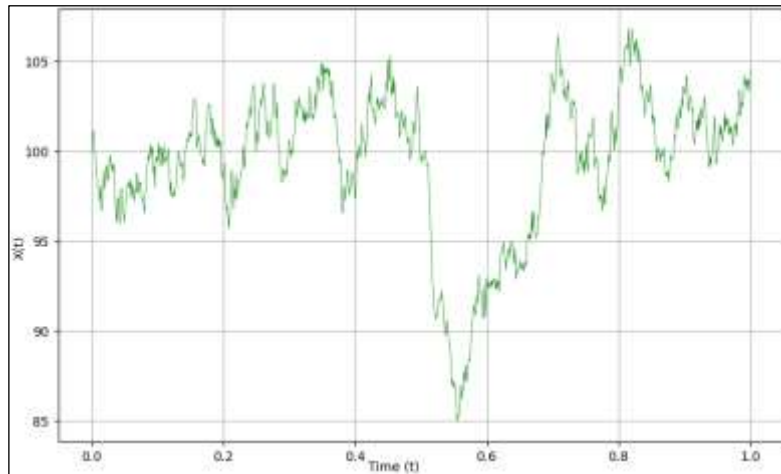


Figure 2: Stochastic Solution of model with negligence of fractional term.

Source: Authors, (2026).

**Comments.** The numerical solutions, however, show a non-monotonic trend: they increase from  $3b1 = 0.1$  (3.1123) to  $3b1 = 0.5$  (3.3451), then drop sharply at  $3b1 = 0.6$  (3.1999) before stabilizing near  $3b1 = 0.9$  (3.2805). This irregularity at  $3b1 = 0.6$  could indicate a computational anomaly, a genuine feature of the system, or the need for a refined numerical approach. The near-convergence of numerical solutions at higher fractional orders (0.7-0.9) suggests potential stability as  $3b1$  approaches 1.0. The lone stochastic solution at  $3b1 = 0.5$  is slightly lower than its numerical counterpart (3.28 vs. 3.3451), hinting at possible methodological differences or approximation errors. For a more robust analysis, the stochastic solutions should be extended across all fractional orders, and the numerical method's accuracy particularly around  $3b1 = 0.6$  should be validated. Additionally, error estimates or graphical representations could further clarify the relationship between these methods. Overall, while the table offers insights, expanding the dataset and investigating discrepancies would strengthen the conclusions.

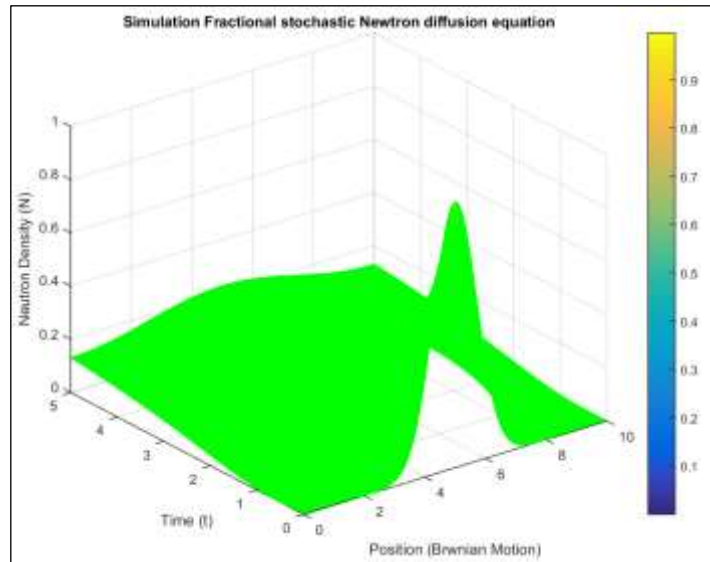


Figure 3: Diffusion flux Newtron.  
Source: Authors, (2026).

In this section, for finite difference approximation, we equally sub-divide the intervals  $[x_0, L]$ .

## VI. CONCLUSION AND FUTURE DIRECTIONS

The generalized stochastic fractional differential solution proves more comprehensive than the standard stochastic solution. Our simulations demonstrate that as the discretization number increases, the Caputo-Hadamard fractional derivative tends toward zero. This indicates that the system converges to a purely stochastic solution, highlighting a key property: the fractional framework encompasses the stochastic case as a special instance. This behavior plays a fundamental role in studying systems where exact solutions are attainable. The standard stochastic solution thus emerges as a specific subset of the broader fractional approach, offering a limited perspective compared to the generalized model. In future studies, we will conduct comparative analyses of solutions derived from different types of fractional derivatives (e.g., Caputo, Caputo Katugampola, Hadamard) and varying fractional orders. Such comparisons will validate a robust, generalized methodology for applications in modern numerical physics. By systematically evaluating these approaches, we aim to identify the most versatile and accurate techniques for modelling complex physical systems, bridging theoretical rigor with practical computational efficiency.

## VII. AUTHOR'S CONTRIBUTION

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